DEFENSE DOCUMENTATION CENTER

FOR

SCIENTIFIC AND TECHNICAL INFORMATION

CAMERON STATION, ALEXANDRIA, VIRGINIA

THE MITRE CORPORATION
P. O. Box 208
Bedford, Massachusetts

$Scale \sim 1$

ON THE SCATTERING OF
PLANE COMPRESSIONAL WAVES
BY A SPHERICAL OBSTACLE

BY

YIH-HSING PAO*
Department of Engineering Mechanics & Materials,
Cornell University, Ithaca, New York

AND

C. C. MOW
MITRE Corporation, Bedford, Massachusetts

MARCH 1962

* Consultant, MITRE Corporation

ABSTRACT

The scattering of plane compressional waves by a spherical obstacle in an elastic solid, which was investigated by Ying and Truell is examined further. For a rigid inclusion, the boundary conditions are redefined to take into consideration the motion of the inclusion inside the solid. By a proper limiting process, it is shown that the solutions for a rigid insert, a fluid sphere, a cavity, or an obstacle in a fluid, are all derivable from the general results of an elastic inclusion. In each case, the rates of energy scattering due to a small obstacle are found to be inversely proportional to the fourth power of wave length.

THE MITRE CORPORATION
P. O. Box 208
Bedford, Massachusetts

$\subset \cap \, \alpha^{'} \, \in \, - \, ^{'}$

ON THE SCATTERING OF
PLANE COMPRESSIONAL WAVES
BY A SPHERICAL OBSTACLE

BY

YIH-HSING PAO[*]
Department of Engineering Mechanics & Materials,
Cornell University, Ithaca, New York

AND

C. C. MOW
MITRE Corporation, Bedford, Massachusetts

MARCH 1962

[*] Consultant, MITRE Corporation

ABSTRACT

The scattering of plane compressional waves by a spherical obstacle in an elastic solid, which was investigated by Ying and Truell is examined further. For a rigid inclusion, the boundary conditions are redefined to take into consideration the motion of the inclusion inside the solid. By a proper limiting process, it is shown that the solutions for a rigid insert, a fluid sphere, a cavity, or an obstacle in a fluid, are all derivable from the general results of an elastic inclusion. In each case, the rates of energy scattering due to a small obstacle are found to be inversely proportional to the fourth power of wave length.

CONTENTS

I. INTRODUCTION

The scattering of a plane compressional wave by a spherical obstacle in an elastic solid infinite in extent has been investigated by Ying and Truell.[1] Three types of obstacles, an elastic sphere, a rigid sphere, and a cavity are discussed; detailed results being given for the case of Rayleigh scattering, i.e., when the size of the obstacle is much smaller than the incident wave length. The solution of the elastic sphere is quite general in the sense that it applies for various values of material constants (Lamé's constants, density etc.) characterizing the scatterer and the surrounding medium. However, Ying & Truell studied the problems of the rigid sphere and a cavity separately because some terms in the general solution are ambiguous when the limiting values of the density of the inclusion are taken. For the same reason, the fourth type of obstacle, a cavity filled with an inviscid fluid, was also treated independently.[2]

Moreover, the scattering by a rigid sphere was calculated with the assumed boundary condition that the displacement due to the combined incident and reflected waves vanishes at the surface of the sphere. This led to the conclusion that the energy scattered by a small rigid sphere per unit time is independent of the incident wave length; this result being quite different from other well known scattering phenomena, i.e., the rate of energy scattering is inversely proportional to the fourth power of wave length.

The same boundary conditions for a rigid obstacle have been applied in many related investigations.[3,4,5] In a recent study on the dynamical stress concentration in an elastic plate with a circular rigid insert,[6] the authors find that if the zero displacement boundary conditions are used, stresses will become infinitely large at certain points in the plate, a conclusion even more drastic than the unusual rate of energy propagation mentioned above.

In a fluid medium, the disturbance of plane waves (sound waves) due to a rigid sphere was first treated by Rayleigh.[7] He showed that for a fixed sphere, there was a net unbalanced force acting on the surface of the sphere, i.e., if the sphere were not held fixed by some external constraint, it would move. A modified solution for an unconstrained rigid sphere was given by Lamb.[8] However, in either case, the rate of energy scattering is inversely proportional to the fourth power of wavelength.

There is, however, a major difference between a movable and a fixed obstacle in an elastic solid. With the view of obtaining a mechanical illustration of the selective absorption of light by a gas. Lamb also investigated the scattering of shear waves in an incompressible solid.[10] If the boundary of the sphere is absolutely fixed, the rate of energy scattering, called by Lamb the "dissipation ratio," is independent of wavelength. Lamb explained that this was accounted for by the abnormal degree of constraint imposed on both radial and tangential displacements of the surrounding medium, whereas in the corresponding fluid problem, there was complete freedom of lateral motion at the surface of the rigid sphere. When the sphere is allowed to move with the surrounding medium, the dissipation ratio conforms with the general rule. Later, Sezawa[9] investigated the problem of scattering of elastic waves.

Nishimura and Jimbo[11] also investigated the diffraction of plane compressional waves by three types of scatterer, i.e., elastic, rigid and cavity. However, as a result of the assumed standing waves, only even terms of series expression for waves in polar coordinates are needed, thus no complication such as that encountered in the case of traveling waves arises.

In this paper, the boundary conditions for rigid scatterer are redefined in order to take into account the rigid body translation.[12] It is then shown that by a limiting process, the results for a rigid sphere, a cavity or even a fluid sphere can all be derived from the general solution of scattering due to an elastic inclusion. The process can be applied further to deduce the results of the scattering of sound in fluid. The rates of energy scattering are found, in all

cases, to be inversely proportional to the fourth power of the incident wave length.

In the interests of brevity, only results which are to be compared with that in References 1, 2 or which are essential to carry out the limiting process are given. For a detailed discussion and references, the reader is referred to the original work by Ying and Truell.

II. BASIC EQUATIONS

In an elastic solid with waves propagating symmetrically about the z axis (Figure 1), the displacement vector in spherical coordinates (r, θ, ϕ) can be expressed in terms of two displacement potentials Φ and Ψ, i.e.,

$$\underset{\sim}{u} = \nabla \Phi + \nabla \times (\underset{\sim}{e}_\phi \, \partial \Psi / \partial \theta) \tag{1}$$

where ∇ is the gradient operator and $\underset{\sim}{e}_\phi$ is a unit vector along ϕ coordinate curve. Each potential can then be shown to satisfy a scalar wave equation

$$c_\alpha^2 \nabla^2 \Phi = \partial^2 \Phi / \partial t^2 \tag{2}$$

$$c_\beta^2 \nabla^2 \Psi = \partial^2 \Psi / \partial t^2 \tag{3}$$

where $c_\alpha = \left[(\lambda + 2\mu)/\rho \right]^{1/2}$, $c_\beta = (\mu/\rho)^{1/2}$ are the velocities of compressional (longitudinal) waves and shear (transverse) waves respectively; λ, μ are Lamé's constants, and ρ the density of the solid.

The stress components are related to the displacements, in dyadic notation, as

$$\underset{\sim}{\tau} = \lambda (\nabla \cdot \underset{\sim}{u}) \underset{\sim}{I} + 2\mu (\nabla \underset{\sim}{u} + \underset{\sim}{u} \nabla) \tag{4}$$

where $\underset{\sim}{\tau}$, $\underset{\sim}{I}$ represent the stress tensor and isotropic tensor respectively. Due to axial symmetry, the displacement u_ϕ as well as the stresses $\tau_{r\phi}$, $\tau_{\theta\phi}$ vanish.

When plane waves impinge on the surface of an inclusion embedded in an infinitely extended elastic solid, two types of waves (compressional and shear) are reflected back into the infinite medium and two waves are refracted into the inclusion. For the convenience of ensuing discussion, the infinite solid is designated as medium 1 and the spherical inclusion as medium 2. All material constants such as λ, μ, ρ etc. in each medium will be distinguished by subscript 1 or 2 accordingly. The potentials, displacements and stresses associated with the incident waves will be designated by the superscript (i); those with the reflected waves by (r) and those with the refracted waves by (f).

III. INCIDENT, REFLECTED AND REFRACTED WAVES

Let the incident plane compressional waves traveling in the positive z direction be represented by two potentials (Figure 1)

$$\begin{cases} \Phi^{(i)} = \Phi_0 e^{i(\alpha_1 z - \omega t)} \\ \Psi^{(i)} = 0 \end{cases} \tag{5}$$

where α_1 (= $2\pi/$ wave length) is the wave number of compressional waves in medium 1, and ω is circular frequency. In order to be a solution of the wave Equation (2), the wave number has the following value

$$\alpha_1^2 = \omega^2 \rho_1 / (\lambda_1 + 2\mu_1) .$$

From (1) it is clear that Φ_0, a constant, has the dimension of (length)^2 and $\alpha_1 \Phi_0$ is the amplitude of the incident wave. In spherical coordinates,[13]

$$\Phi^{(i)} = \Phi_0 \sum_{n=0}^{\infty} (2n + 1) i^n j_n(\alpha_1 r) P_n(\cos \theta), \tag{6}$$

in which

$$j_n(x) = \sqrt{\frac{\pi}{2x}} \, J_{n+1/2}(x)$$

is the spherical Bessel function of the first kind, $P_n(x)$ is the Legendre polynomials. In Equation (6) and sequel the time factor $e^{-i\omega t}$ is omitted whenever its occurrence is apparent. By substituting (5) or (6) into (1) and (4), the displacements and stresses of the incident waves can be determined.

Expressions for the reflected and refracted waves can be obtained from the solutions of the wave Equations (2), (3). In an infinite solid, only waves propagating outward from the center will be considered. Thus the reflected waves are given by

$$\Phi^{(r)} = \sum_{n=0}^{\infty} A_n \, h_n(\alpha_1 r) \, P_n(\cos\theta) \tag{7}$$

$$\Psi^{(r)} = \sum_{n=0}^{\infty} B_n \, h_n(\beta_1 r) \, P_n(\cos\theta) \tag{8}$$

with

$$h_n(x) = h_n^{(1)}(x) = \sqrt{\frac{\pi}{2x}} \, H_{n+1/2}^{(1)}(x)$$

being the spherical Hankel function of first kind. As there is no confusion with the Hankel function of second kind which represents converging wave in this study, the conventional superscript (1) is omitted.

The refracted waves, being confined in the spherical scatterer, are standing waves. They can be represented by

$$\Phi^{(f)} = -\sum_{n=0}^{\infty} C_n j_n(\alpha_2 r) P_n(\cos\theta) \tag{9}$$

$$\Psi^{(f)} = -\sum_{n=0}^{\infty} D_n j_n(\beta_2 r) P_n(\cos\theta) \tag{10}$$

In (7) — (10),

$$\alpha_i^2 = \omega^2 \rho_i/(\lambda_i + 2\mu_i)$$
$$\beta_i^2 = \omega^2 \rho_i/\mu_i, \qquad i = 1, 2 \tag{11}$$

and A_n, B_n, C_n, D_n are coefficients to be determined from the boundary conditions at the surface of the inclusion.

In the medium 1, the resultant waves are then determined by superposing the incident and reflected waves whereas the refracted waves are the only ones in medium 2. Displacements and stresses in each medium are summarized below.

$$\Phi_1 = \Phi^{(i)} + \Phi^{(r)},$$
$$\Psi_1 = \Psi^{(i)} + \Psi^{(r)}, \tag{12}$$

$$u_{r1} = \frac{1}{r}\sum_{n=0}^{\infty}\left[-\Phi_0\,\delta_1 + A_n\,\delta_{11} + B_n\,\delta_{12}\right] P_n(\cos\theta)$$

$$\tag{13}$$

$$u_{\theta1} = \frac{1}{r}\sum_{n=0}^{\infty}\left[-\Phi_0\,\delta_2 + A_n\,\delta_{21} + B_n\,\delta_{22}\right]\frac{dP_n(\cos\theta)}{d\theta},$$

$$\tau_{rr1} = \frac{2\mu_1}{r^2} \sum_{n=0}^{\infty} \left[-\Phi_0 \, \delta_3 + A_n \, \delta_{31} + B_n \, \delta_{32} \right] \, P_n(\cos \theta) \quad ,$$

(14)

$$\tau_{r\theta 1} = \frac{2\mu_1}{r^2} \sum_{n=0}^{\infty} \left[-\Phi_0 \, \delta_4 + A_n \, \delta_{41} + B_n \, \delta_{42} \right] \frac{dP_n(\cos \theta)}{d\theta} \, ,$$

$$\Phi_2 = \Phi^{(f)}, \qquad \Psi_2 = \Psi^{(f)} \, ,$$ (15)

$$u_{r2} = -\frac{1}{r} \sum_{n=0}^{\infty} \left[C_n \delta_{13} + D_n \, \delta_{14} \right] P_n(\cos \theta)$$

(16)

$$u_{\theta 2} = -\frac{1}{r} \sum_{n=0}^{\infty} \left[C_n \, \delta_{23} + D_n \, \delta_{24} \right] \frac{dP_n(\cos \theta)}{d\theta} \, ,$$

$$\tau_{rr2} = -\frac{2\mu_2}{r^2} \sum_{n=0}^{\infty} \left[C_n \delta_{33} + D_n \, \delta_{34} \right] P_n(\cos \theta)$$

(17)

$$\tau_{r\theta 2} = -\frac{2\mu_2}{r^2} \sum_{n=0}^{\infty} \left[C_n \, \delta_{43} + D_n \, \delta_{44} \right] \frac{dP_n(\cos \theta)}{d\theta} \, ,$$

where

$$\delta_1 = -i^n (2n+1) \left[n j_n(\alpha_1 r) - \alpha_1 r j_{n+1}(\alpha_1 r) \right]$$

$$\delta_2 = -i^n (2n+1) \, j_n(\alpha_1 r)$$

$$\delta_3 = -i^n (2n+1) \left[(n^2 - n - \tfrac{1}{2} \beta_1^2 r^2) \, j_n(\alpha_1 r) + 2\alpha_1 r \, j_{n+1}(\alpha_1 r) \right]$$

$$\delta_4 = -i^n (2n+1) \left[(n-1) \, j_n(\alpha_1 r) - \alpha_1 r j_{n+1}(\alpha_1 r) \right] \; ,$$

$$\delta_{11} = n h_n(\alpha_1 r) - \alpha_1 r \, h_{n+1}(\alpha_1 r)$$

$$\delta_{21} = h_n(\alpha_1 r)$$

$$\delta_{31} = (n^2 - n - \tfrac{1}{2} \beta_1^2 r^2) \, h_n(\alpha_1 r) + 2\alpha_1 r h_{n+1}(\alpha_1 r)$$

$$\delta_{41} = (n-1) h_n(\alpha_1 r) - \alpha_1 r h_{n+1}(\alpha_1 r) \; ,$$

$$\delta_{12} = -n(n+1) h_n(\beta_1 r)$$

$$\delta_{22} = -(n+1) h_n(\beta_1 r) + \beta_1 r h_{n+1}(\beta_1 r)$$

$$\delta_{32} = -n(n+1) \left[(n-1) h_n(\beta_1 r) - \beta_1 r h_{n+1}(\beta_1 r) \right] \tag{18}$$

$$\delta_{42} = -(n^2 - 1 - \tfrac{1}{2} \beta_1^2 r^2) h_n(\beta_1 r) - \beta_1 r h_{n+1}(\beta_1 r) \; ,$$

$$\delta_{13} = n j_n(\alpha_2 r) - \alpha_2 r j_{n+1}(\alpha_2 r)$$

$$\delta_{23} = j_n(\alpha_2 r)$$

$$\delta_{33} = (n^2 - n - \tfrac{1}{2} \beta_2^2 r^2) \, j_n(\alpha_2 r) + 2\alpha_2 r j_{n+1}(\alpha_2 r)$$

$$\delta_{43} = (n-1) j_n(\alpha_2 r) - \alpha_2 r j_{n+1}(\alpha_2 r) \; ,$$

$$\delta_{14} = -n(n+1) j_n(\beta_2 r)$$

$$\delta_{24} = -(n+1) j_n(\beta_2 r) + \beta_2 r j_{n+1}(\beta_2 r)$$

$$\delta_{34} = -n(n+1) \left[(n-1) j_n(\beta_2 r) - \beta_2 r j_{n+1}(\beta_2 r) \right]$$

$$\delta_{44} = -(n^2 - 1 - \tfrac{1}{2} \beta_2^2 r^2) \, j_n(\beta_2 r) - \beta_2 r j_{n+1}(\beta_2 r) \; .$$

Note that $\delta_{13} = \Re e \ (\delta_{11})$ and $\delta_{14} = \Re e \ (\delta_{12})$ provided the subscripts of wave numbers α_i, β_i are changed properly. All δ's are dimensionless numbers and A_n etc. have the same dimensions as Φ_0.

IV. ELASTIC INCLUSION

If the spherical inclusion of radius a, is elastic, and it is bound to the surrounding medium, the tractions and displacements must be continuous at the interface. Thus at r = a, the continuity conditions are

$$u_r^{(i)} + u_r^{(r)} = u_r^{(f)}$$

$$u_\theta^{(i)} + u_\theta^{(r)} = u_\theta^{(f)}$$

$$\tau_{rr}^{(i)} + \tau_{rr}^{(r)} = \tau_{rr}^{(f)}$$

$$\tau_{r\theta}^{(i)} + \tau_{r\theta}^{(r)} = \tau_{r\theta}^{(f)}$$

(19)

Since the Legendre polynomials $P_n(\cos \theta)$ and the associated Legendre polynomials

$$P_n^1 (\cos \theta) = - \frac{d}{d\theta} P_n(\cos \theta)$$

each form an orthogonal complete set, when (13), (14), (16) and (17) are substituted into Equations (19), the coefficients of the Legendre polynomials on both sides of (19) must be equal for each value of n. This results in four simultaneous algebraic equations, sufficing to determine the four unknown coefficients A_n, B_n, C_n, and D_n. In matrix form, these equations are

$$\begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} \\ E_{21} & E_{22} & E_{23} & E_{24} \\ E_{31} & E_{32} & pE_{33} & pE_{34} \\ E_{41} & E_{42} & pE_{43} & pE_{44} \end{bmatrix} \begin{bmatrix} A_n \\ B_n \\ C_n \\ D_n \end{bmatrix} = \Phi_0 \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{bmatrix}$$

(20)

where

$$E_{ij} = (\delta_{ij})_{r=a}$$

and

$$p = \mu_2/\mu_1 \qquad (21)$$

Except for the limiting cases when the arguments of the spherical Bessel functions become very small or very large, these equations can best be solved numerically.

The scattering of waves by small obstacles ($\alpha_1 a \ll 1$) is generally known as Rayleigh scattering. From (11) it follows that if $\alpha_1 a \ll 1$, $\alpha_2 a$ and $\beta_i a$ will all be very small. Thus the spherical Bessel and Hankel functions in (20) can be approximated by the leading terms of their series expansions which are

$$j_n(x) = (2x)^n \sum_{r=0}^{\infty} \frac{(-1)^r (n+r)!}{r!\,(2n+2r+1)!}\, x^{2r}$$

$$h_n(x) = \frac{e^{ix}}{x^{n+1}} \sum_{r=0}^{\infty} \frac{(-i)^{n-r+1}(n+r)!}{r!\,(n-r)!\,2^r}\, x^{n-r}$$

Thus for the case of Rayleigh scattering,

$$A_0 = -\frac{1}{3}i\Phi_0\left[1 - \frac{3(\beta_1/\alpha_1)^2}{3p(\beta_2/\alpha_2)^2 - 4(p-1)}\right](\alpha_1 a)^3,$$

$$A_1 = \frac{1}{3}\Phi_0(1 - \rho_2/\rho_1)(\alpha_1 a)^3,$$

$$A_n = i^{n-1}\Phi_0\left[\frac{2^n n!}{(2n)!}\right]^2 N_p(\alpha_1 a)^{2n-1}, \qquad n > 1, \tag{22}$$

$$N_p = \frac{(4n^2 - 1)(p - 1)}{p - 1 + \frac{1}{n}\left[(n+1)p + \frac{2n^2 + 1}{2n - 2}\right]\left(\frac{\beta_1}{\alpha_1}\right)^2}$$

$$B_n = -(\beta_1/\alpha_1)^{n+1}A_n/n \qquad\qquad n > 0 \;.$$

The Coefficient B_0 is not needed in the calculation of stress and displacements.
If Φ_0 is set equal to $(-1)^n i/\alpha_1$, the above results agree with those of Reference 1.
It is worth to note here that

$$\left(\frac{\beta_i}{\alpha_i}\right)^2 = \frac{\lambda_i + 2\mu_i}{\mu_i} = \frac{2(1 - \sigma_i)}{1 - 2\sigma_i} \qquad i = 1,\, 2$$

where σ_i are the Poissons's ratios of the media.

V. FLUID INCLUSION

An inviscid fluid will neither resist shearing stress nor support shear waves. If a cavity in an elastic solid is filled with such a fluid, only compressional waves are refracted into the fluid. The boundary conditions at $r = a$ are reduced from (19) to

$$u_r^{(i)} + u_r^{(r)} = u_r^{(f)}$$

$$\tau_{rr}^{(i)} + \tau_{rr}^{(r)} = \tau_{rr}^{(f)} \tag{23}$$

$$\tau_{r\theta}^{(i)} + \tau_{r\theta}^{(r)} = 0$$

The circumferential displacement u_θ in the solid might be different from that in the fluid at the interface. Such a paradox in discontinuity is due to the assumed zero viscosity.

The displacements and stresses in the elastic solid are obviously still the same as (13), (14) with A_n, B_n satisfying (23). In the fluid, they can be obtained from (16), (17) by taking the limiting values as $\mu_2 \to 0$.

It should be noted from (11) that as $\mu_2 \to 0$, $\beta_2 \to \infty$; but

$$\mu_2 \beta_2^2 = \omega^2 \rho_2 = \mu_1 \beta_1^2 \rho_2 / \rho_1 \tag{24}$$

remains finite. By setting $\mu_2 = 0$, $D_n = 0$ (no refracted shear wave), and taking

$$\mu_2 \mathcal{E}_{33} \to \mu_1 \mathcal{E}_{33}^f ,$$

with

$$\mathcal{E}_{33}^f = -\frac{1}{2} (\rho_2/\rho_1) \beta_1^2 r^2 j_n(\alpha_2 r) \tag{25}$$

one finds that the boundary conditions (23) require that A_n, B_n, C_n satisfy three equations, i.e.,

$$\begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{31} & E_{32} & E_{33}^f \\ E_{41} & E_{42} & 0 \end{bmatrix} \begin{bmatrix} A_n \\ B_n \\ C_n \end{bmatrix} = \Phi_0 \begin{bmatrix} E_1 \\ E_3 \\ E_4 \end{bmatrix} \qquad (26)$$

These equations agree with that given by Einspruch and Truell.[2]

The same equations can directly be deduced from the case of an elastic inclusion. Recall that $\beta_2 \to \infty$ as $\mu_2 \to 0$; hence the spherical Bessel functions with argument $\beta_2 a$ in the matrix (20) can be approximated by the asymptotic formula

$$j_n(x) \underset{x \to \infty}{\sim} \frac{1}{x} \cos\left(x - \frac{n+1}{2}\pi\right) .$$

It is clear that as $\beta_2 a \to \infty$,

$$j_n(\beta_2 a) = o(\epsilon)$$

$$\beta_2 a\, j_{n+1}(\beta_2 a) = o(\epsilon^0)$$

$$\mu_2 = o(\epsilon^2)$$

where ϵ is an infinitesimal quantity and "$o(\epsilon)$" means "the order of magnitude of ϵ". Applying these limits to all elements of the matrix, one finds that

$$E_{14} = o(\epsilon), \qquad pE_{34} = o(\epsilon^2),$$

$$pE_{44} = o(\epsilon), \qquad pE_{43} = o(\epsilon^2),$$

$$E_{24} = o(\epsilon^0), \qquad pE_{33} \to E_{33}^f .$$

By neglecting all elements with order of magnitude higher than ϵ^0, Equations (20) reduce to

$$
\begin{bmatrix}
E_{11} & E_{12} & E_{13} & 0 \\
E_{21} & E_{22} & E_{33} & 0(\epsilon^0) \\
E_{31} & E_{32} & E_{33}^f & 0 \\
E_{41} & E_{42} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
A_n \\
B_n \\
C_n \\
D_n
\end{bmatrix}
= \Phi_0
\begin{bmatrix}
E_1 \\
E_2 \\
E_3 \\
E_4
\end{bmatrix}
\qquad (27)
$$

For the determination of the coefficients A_n, B_n, and C_n, these equations can easily be reduced to (26).

For Rayleigh scattering, A_n, B_n can be directly deduced from (22) by taking the limiting values as $p \to 0$, $\beta_2 \to \infty$. They have the following values which check with those given by Einspruch and Truell.[2,14]

$$
A_0 = -\frac{1}{3} i \Phi_0 \left[1 - \frac{3(\beta_1/\alpha_1)^2}{3\rho_2 \beta_1^2/\rho_1 \alpha_2^2 + 4} \right] (\alpha_1 a)^3 ,
$$

$$
A_1 = \frac{1}{3} \Phi_0 (1 - \rho_2/\rho_1)(\alpha_1 a)^3 ,
$$

$$
A_n = i^{n-1} \Phi_0 \left[\frac{2^n n!}{(2n)!} \right]^2 N_p (\alpha_1 a)^{2n-1} , \quad n > 1 \qquad (28)
$$

$$
N_p = \frac{4n^2 - 1}{1 - \dfrac{2n^2 + 1}{n(2n-2)} \left(\dfrac{\beta_1}{\alpha_1} \right)^2} ,
$$

$$
B_n = - (\beta_1/\alpha_1)^{n+1} A_n/n , \qquad n > 0 .
$$

VI. RIGID SPHERE

1. Boundary Conditions

When the inclusion is much more rigid than the surrounding material, it may be treated as a perfectly rigid sphere. By definition, the distance between any two points in a rigid body remains constant at all times, but because of the surrounding medium being elastic, the sphere will translate as a rigid body under the impact of the incident waves. Unless the sphere is fixed in position by external forces or other means, which is hardly conceivable when it is embedded in an infinite solid, the displacements of the elastic medium should be equal to the translation of the inclusion at the interface.

Let U_z denote the rigid body motion of the sphere along the direction of incident waves (z axis). The boundary conditions at $r = a$ become

$$u_r^{(i)} + u_r^{(r)} = U_z \cos \theta = U_z P_1(\cos \theta)$$

$$u_\theta^{(i)} + u_\theta^{(r)} = - U_z \sin \theta = U_z \frac{dP_1(\cos \theta)}{d\theta}$$

(29)

The translation U_z is to be determined from the equation of motion

$$m \ddot{U}_z = \iint (\tau_{rr} \cos \theta - \tau_{r\theta} \sin \theta) a^2 \sin \theta \, d\theta \, d\phi$$

(30)

where $m = \frac{4}{3} \pi a^3 \rho_2$ being the mass of the sphere and the integral over the spherical surface representing the force component acted on the sphere by the surrounding medium.

Restoring the time factor and substituting (14) into (30), one finds[15]

$$U_z = \frac{\eta}{a} \left[3i \, \Phi_0 \, j_1(\alpha_1 a) + A_1 h_1(\alpha_1 a) - 2B_1 h_1(\beta_1 a) \right] e^{-i\omega t} \tag{31}$$

with

$$\eta = \rho_1/\rho_2 \, .$$

Owing to the orthogonality of $P_n(\cos\theta)$ and $P_n^1(\cos\theta)$, the terms with $n \neq 1$ in the series (14) vanish after integration.

The boundary conditions used in Reference 1 and others are special cases of (29) with $\eta = 0$, which implies that the density of the sphere is infinitely large. However no consideration of rigid body motion is needed if only standing waves are involved as in Nishimura and Jimbo's work.[11] In that case, the sphere, even with finite density, is always being held in the equilibrium position by two trains of waves propagating in opposite directions.

2. Solution

For $n \neq 1$, the coefficients A_n, B_n are determined from the matrix equations

$$\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} \begin{bmatrix} A_n \\ B_n \end{bmatrix} = \Phi_0 \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \qquad n \neq 1 \tag{32}$$

Coefficients A_1, B_1 are given by a different set of equations

$$\begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \Phi_0 \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \tag{33}$$

with

$$e_{11} = (1 - \eta)\, h_1(\alpha_1 a) - \alpha_1 a h_2(\alpha_1 a)$$

$$e_{21} = (1 - \eta)\, h_1(\alpha_1 a)$$

$$e_{12} = -2(1 - \eta)\, h_1(\beta_1 a)$$

$$e_{22} = -2(1 - \eta)\, h_1(\beta_1 a) + \beta_1 a h_2(\beta_1 a) \tag{34}$$

$$e_1 = -3i\left[(1 - \eta)\, j_1(\alpha_1 a) - \alpha_1 a j_2(\alpha_1 a) \right]$$

$$e_2 = -3i\, (1 - \eta)\, j_1(\alpha_1 a)$$

If the scatterer is very small in comparison with the incident wave length, the coefficients are found as

$$A_0 = -\frac{1}{3} i \Phi_0 (\alpha_1 a)^3$$

$$A_1 = \frac{1}{3} \Phi_0 (1 - \rho_2/\rho_1)(\alpha_1 a)^3$$

$$A_n = i^{n-1} \Phi_0 \left[\frac{2^n n!}{(2n)!} \right]^2 N_p (\alpha_1 a)^{2n-1} \qquad n > 1 \tag{35}$$

$$N_p = \frac{n(4n^2 - 1)}{n + (n + 1)(\beta_1/\alpha_1)^2}$$

$$B_n = -(\beta_1/\alpha_1)^{n+1} A_n/n \qquad n > 0$$

A_n and B_n are identical with those given in Reference 1 except for $n = 1$.

3. Derivation from Solutions of Elastic Sphere

The lengthy calculations of the rigid body translation given in (31) can be·· avoided if the solution of an elastic inclusion is known in advance. Caution should be exercised however in deriving the results from Section IV. Although it is well known that no waves propagate in a rigid solid, one cannot simply set all C_n, D_n of the refracted waves to be zero in this study.

In (16) or (13) for constant r, there are two terms in the series having the form

$$u_r = b_1 P_1(\cos \theta) = b_1 \cos \theta$$

$$u_\theta = -c_1 \left[dP_1(\cos \theta)/d\theta \right] = c_1 \sin \theta$$

(36)

where b_1, c_1 are constants. By transformation, the corresponding displacements in z - q coordinates (Figure 1) can be shown as

$$u_z = \frac{1}{2}(b_1 - c_1) + \frac{1}{2}(b_1 + c_1) \cos 2\theta$$

$$u_q = \frac{1}{2}(b_1 + c_1) \sin 2\theta$$

There is clearly a rigid translation $\frac{1}{2}(b_1 - c_1)$ in the z - direction implied in (36). Indeed, this rigid body translation can be separated from the deformation if (36) is rewritten in the following form[16]

$$u_r = \frac{1}{2}(b_1 + c_1) \cos \theta + \frac{1}{2}(b_1 - c_1) \cos \theta$$

(36a)

$$u_\theta = \frac{1}{2}(b_1 + c_1) \sin \theta - \frac{1}{2}(b_1 - c_1) \sin \theta$$

By suppressing all terms (including n = 1) in the series expressions for $u_r^{(f)}$, $u_\theta^{(f)}$, the sphere is then assumed to be rigid and fixed in space. For a movable sphere, this simple suppression will lead to erroneous result. In fact, the second terms in (36a) are exactly what have been incorporated in setting up the boundary conditions (29).

A limiting process analogous to that in Section V will now be used to derive the rigid sphere solution. If the medium 2 is perfectly rigid, it is seen that

$$\lambda_2 \rightarrow \infty, \qquad \mu_2 \rightarrow \infty$$

but ρ_2 remains finite. Thus by taking only the leading terms of the series of spherical Bessel functions, one finds

$$\alpha_2 a = o(\epsilon), \qquad \beta_2 a = o(\epsilon),$$

$$p = o(\epsilon^{-2}),$$

$$j_n(\alpha_2 a) = o(\epsilon^n), \qquad \beta_2 a j_{n+1}(\beta_2 a) = o(\epsilon^{n+2}).$$

If these limiting values are substituted into the matrix (20), it can be shown that for $n \neq 1$;

$$E_{i\ell} = o(\epsilon^n), \quad i = 1, 2; \quad \ell = 3, 4$$

$$pE_{k\ell} = o(\epsilon^{n-2}), \quad k, \ell = 3, 4$$

for n = 1,

$$E_{i\ell} = o(\epsilon),$$

$$pE_{k\ell} = o(\epsilon),$$

It follows then, for $n \neq 1$, the matrix reduces to (32) when $E_{i\ell}$ are neglected in comparison with $pE_{k\ell}$. For n = 1, the complete 4 x 4 matrix should be used

with the spherical Bessel functions in the third and fourth columns being replaced by the corresponding small argument approximations. Coefficients A_1, B_1 thus determined are found to be the same as that from Equations (33). Furthermore, all coefficients for the case of Rayleigh scattering as given in (35) can directly be deduced from (22) by letting p approach infinity.

Although in this limiting process, C_n, D_n are treated as finite numbers, the refracted waves still vanish because of the small values for $\beta_2 a$, $\alpha_2 a$ which appear in the series representations of Φ_2, Ψ_2.

VII. SPHERICAL CAVITY

Unlike a rigid inclusion, the case of spherical cavity presents no complication in analysis. Coefficients A_n, B_n are fixed by the boundary conditions at $r = a$,

$$
\tau_{rr}^{(i)} + \tau_{rr}^{(r)} = 0
$$

$$
\tau_{r\theta}^{(i)} + \tau_{r\theta}^{(r)} = 0 ,
$$

(37)

which yields two simultaneous equations

$$
\begin{bmatrix} E_{31} & E_{32} \\ E_{41} & E_{42} \end{bmatrix} \begin{bmatrix} A_n \\ B_n \end{bmatrix} = \Phi_0 \begin{bmatrix} E_3 \\ E_4 \end{bmatrix}
$$

(38)

These equations can also be derived from (20) by a limiting process ($\lambda_2 \rightarrow 0$, $\mu_2 \rightarrow 0$, $\rho_2 \rightarrow 0$) similar to that used in Section V.

With small $\alpha_1 a$, A_n and B_n can be obtained by either solving Equation (38) or by taking $p \rightarrow 0$, $\rho_2 \rightarrow 0$ in (22). They are listed below for reference.

$$A_0 = -\frac{1}{3} i \Phi_0 \left[1 - \frac{3}{4} \left(\frac{\beta_1}{\alpha_1} \right)^2 \right] (\alpha_1 a)^3$$

$$A_1 = \frac{1}{3} \Phi_0 (\alpha_1 a)^3$$

$$A_n = i^{n-1} \Phi_0 \left[\frac{2^n n!}{(2n)!} \right]^2 N_p (\alpha_1 a)^{2n-1} \qquad n > 1 \qquad (39)$$

$$N_p = \frac{4n^2 - 1}{1 - \frac{2n^2 + 1}{2n(n-1)} \left(\frac{\beta_1}{\alpha_1} \right)^2}$$

$$B_n = - (\beta_1/\alpha_1)^{n+1} A_n/n \qquad n > 0$$

VIII. SCATTERER IN FLUID

By applying an analogous limiting process of reducing the shear rigidity of medium 1 $(\mu_1 \to 0)$ to the solutions given hitherto, the scattering of sound (compressional) waves in a nonviscous, compressible fluid can be obtained. The obstacle could be an elastic sphere, a rigid sphere (fixed or movable), a cavity (bubble in liquid) or an acqueous sphere (hydrosol). Most of these solutions have long been in existence. It is sufficient here, as an example, only to discuss the scattering of sound by a rigid sphere, which was first treated by Rayleigh[7] and Lamb.[8]

Because in the inviscid fluid, no shear wave can propagate, the shear modulus $\mu_1 \to 0$ and the shear wave number $\beta_1 \to \infty$. By the application of the asymptotic formula of spherical Hankel functions for large argument

$$h_n(x) \underset{x \to \infty}{\sim} (-i)^{n+1} \frac{e^{ix}}{x},$$

it can be shown that, in (32),

$$\beta_1 a = o(\epsilon^{-1}), \qquad h_n(\beta_1 a) = o(\epsilon);$$

$$E_{12} = o(\epsilon)$$

$$E_{22} = o(\epsilon^0)$$

If E_{12} is neglected in comparison with E_{22}, Equation (32) degenerates into,

$$A_n E_{11} = \Phi_0 E_1 \tag{40}$$

and

$$A_n = -\Phi_0 i^n (2n+1) \frac{n j_n(\alpha_1 a) - \alpha_1 a j_{n+1}(\alpha_1 a)}{n h_n(\alpha_1 a) - \alpha_1 a h_n(\alpha_1 a)} \tag{40a}$$

This can be compared with Rayleigh's result for a fixed rigid sphere, which is obtained from the boundary condition

$$u_r^{(i)} + u_r^{(r)} = 0 \tag{41}$$

For a movable sphere, Equation (40a) is still valid except for n = 1. The equation for A_1 should be deduced from (33) by applying the same limiting process with the result

$$A_1 = -3 i \phi_0 \frac{(1-\eta) j_1(\alpha_1 a) - \alpha_1 a j_2(\alpha_1 a)}{(1-\eta) h_1(\alpha_1 a) - \alpha_1 a h_2(\alpha_1 a)} \tag{42}$$

Equation (42) can be converted to the results given by Lamb[8] who first took into consideration the rigid translation of the sphere.

For viscous fluid, there is a simple analogy between waves in an elastic solid and a viscous fluid. A complete discussion has been given by Epstein.[17]

IX. SCATTERING CROSS SECTION

The scattering cross section is defined as the ratio of the total energy scattered per unit time by the obstacle to the energy per unit area carried per unit time by the incident wave. In the present case, the total energy propagated across a spherical surface of radius b, concentric with the scatterer is [1,2]

$$Q^{(r)} = - \frac{i\omega}{2} \iint \left[\tau_{rr} u_r^* + \tau_{r\theta} u_\theta^* - \tau_{rr}^* u_r - \tau_{r\theta}^* u_\theta \right] b^2 \sin\theta \, d\theta d\phi \qquad (43)$$

The energy carried by the incident wave per unit area is

$$q^{(i)} = - \frac{i\omega}{2A} \iint_A (\tau_{zz} u_z^* - \tau_{zz}^* u_z) \, dxdy \qquad (44)$$

The superscript (r) and (i) of the stresses and displacements in (43), (44) respectively are omitted, and an asterisk indicates complex conjugates. With the values of τ_{ij}, u_j in (13) and (14) substituted into these equations the scattering cross section is found to be

$$\gamma = \frac{Q^{(r)}}{q^{(i)}} = 4\pi \sum_{n=0}^{\infty} \frac{1}{2n+1} \left[\frac{1}{\alpha_1^2} \left| \frac{A_n}{\Phi_0} \right|^2 + n(n+1) \frac{1}{\alpha_1 \beta_1} \left| \frac{B_n}{\Phi_0} \right|^2 \right] \qquad (45)$$

It follows then γ has a dimension of an area, hence the name "cross section."

By comparing the coefficients A_n, B_n with various n's given by (22), (28), (35) and (39), it is seen that only the ones of n = 0, 1, 2 have the same order of magnitude for small $\alpha_1 a$. With different kinds of spherical obstacles in an elastic solid, the scattering cross sections all can be expressed, with the terms of order higher than α_1^4 neglected, as

$$\gamma \ = \ (4\pi/9)g \ a^2 (\alpha_1 a)^4, \tag{46}$$

in which g takes the following values:

(1)   Elastic Sphere (λ_1, μ_1, ρ_1; λ_2, μ_2, ρ_2 all finite)

$$g = \left[1 - \frac{3(\beta_1/\alpha_1)^2}{3p(\beta_2/\alpha_2)^2 - 4p + 4} \right]^2 + \frac{1}{3}\left[1 + 2\left(\frac{\beta_1}{\alpha_1}\right)^{\frac{3}{2}} \right]\left(1 - \frac{\rho_2}{\rho_1} \right)^2$$

$$+ \ 40 \left[2 + 3\left(\frac{\beta_1}{\alpha_1}\right)^{\frac{5}{2}} \right]\left[\frac{p-1}{4p - 4 + (6p+9)(\beta_1/\alpha_1)^2} \right]^2 \ ,$$

(2)   Fluid Sphere (p = $\mu_2/\mu_1 \to 0$)

$$g = \left[1 - \frac{3(\beta_1/\alpha_1)^2}{3\rho_2\beta_1^2/\rho_1\alpha_2^2 + 4} \right]^2 + \frac{1}{3}\left[1 + 2\left(\frac{\beta_1}{\alpha_1}\right)^{\frac{3}{2}} \right]\left(1 - \frac{\rho_2}{\rho_1} \right)^2$$

$$+ \ 40 \left[2 + 3\left(\frac{\beta_1}{\alpha_1}\right)^{\frac{5}{2}} \right]\left[4 - 9\left(\frac{\beta_1}{\alpha_1}\right)^2 \right]^{-2}$$

(3)   Rigid Sphere (p = $\mu_2/\mu_1 \to \infty$)

$$g = 1 + \frac{1}{3}\left[1 + 2\left(\frac{\beta_1}{\alpha_1}\right)^{\frac{3}{2}} \right]\left(1 - \frac{\rho_2}{\rho_1} \right)^2$$

$$+ \ 40 \left[2 + 3\left(\frac{\beta_1}{\alpha_1}\right)^{\frac{5}{2}} \right]\left[4 + 6\left(\frac{\beta_1}{\alpha_1}\right)^2 \right]^{-2} \ ,$$

(4)   Cavity (p → 0, ρ_2 → 0)

$$g = \left[1 - \frac{3}{4} \left(\frac{\beta_1}{\alpha_1} \right)^2 \right]^2 + \frac{1}{3} \left[1 + 2 \left(\frac{\beta_1}{\alpha_1} \right)^{\frac{3}{2}} \right]$$

$$+ 40 \left[2 + 3 \left(\frac{\beta_1}{\alpha_1} \right)^{\frac{5}{2}} \right] \left[4 - 9 \left(\frac{\beta_1}{\alpha_1} \right)^2 \right]^{-2} .$$

Since q is a dimensionless quantity, it can be concluded that the scattering cross sections are all proportional to the sixth power of the radius of the obstacle, and inversely proportional to the fourth power of the incident wave length.
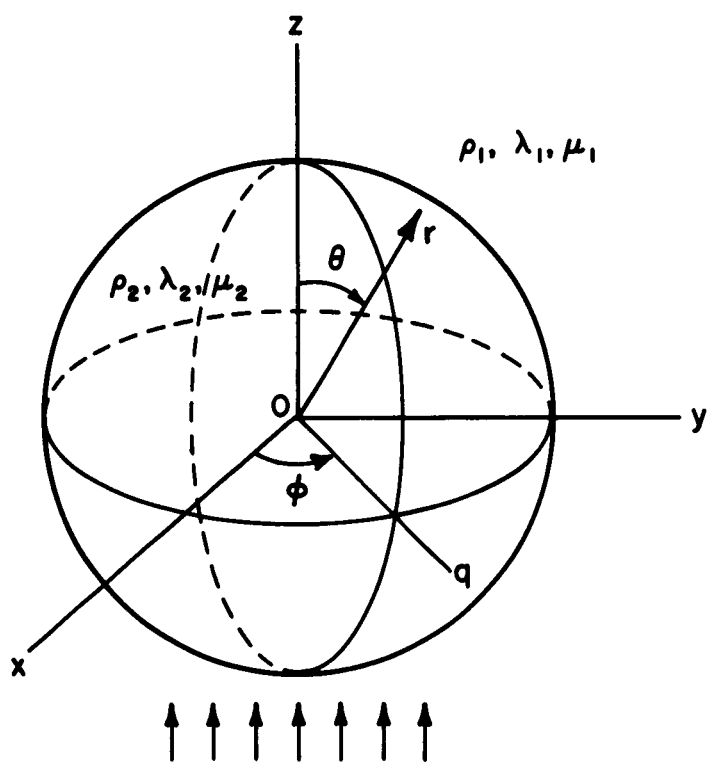
Figure 1

FOOTNOTES AND REFERENCES

1. Ying, C. F. and R. Truell, J. Appl. Phys., V 20, p 1086 (1956).

2. Einspruch, N. G. and R. Truell, J. Acoust. Soc. Am., V 32, p 214 (1960).

3. Knopoff, L., Geophysics, V 34, p 30 (1959).

4. Einspruch, N. G., E. J. Witterholt and R. Truell, J. Appl. Phys., V 31, p 806 (1960).

5. Harumi, K., J. Appl. Phys., V 32, p 1488 (1961).

6. Pao, Y. H. and C. C. Mow, "Dynamical Stress Concentrations in an Elastic Plate with Circular, Rigid Inclusion," Report No. SR 41, MITRE Corporation (1961).

7. Lord Rayleigh, "The Theory of Sound" (Dover Publications, New York, 1945), Sect. 334.

8. Lamb, H., "Hydrodynamics" (Dover Publications, New York, 1945), p 514.

9. Sezawa, K., Bulletin of the Earth Quake Research Institute, Tokyo Imperial University, Japan, V 3, pp 19-41 (1927).

10. Lamb, H., Proc. London Math. Soc., V 32, p 120 (1900).

11. Nishimura, G. and Y. Jimbo, J. Faculty of Engineering, U of Tokyo (Japan), V 24, p 101 (1955).

12. For the case of transverse wave, rotation as well as translation of sphere should be considered. See Reference 10.

13. Morse, P. M. and H. Feshbach, "Methods of Theoretical Physics" (McGraw Hill Book Co., New York, 1953), p 1466.

14. For the purpose of comparison, Φ_0 in (28) should be changed to $(-1)^{n+1}$.

15. This rigid translation has been discussed in Reference 9.

FOOTNOTES AND REFERENCES (Continued)

16. Baron, M. L. and R. Parnes, "Displacements and Velocities Produced by the Diffraction of a Pressure Wave by a Cylindrical Cavity in an Elastic Medium," Paper No. 61-WA-60, Am. Soc. Mechanical Engineers, New York (1961).

17. Epstein, P. S., "Theodore von Karman Anniversary Volume" (California Inst. Technology, California, 1941) p 162.